

CENTRAL FIGURE-8 CROSS-CUTS MAKE SURFACES CYLINDRICAL

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ABSTRACT. We prove: *If a complete connected C^2 surface M in \mathbf{R}^3 has general position, intersects some plane along a clean figure-8 (a loop with total curvature zero) and all compact intersections with planes have central symmetry, then M is a (geometric) cylinder over some central figure-8.* On the way, we establish interesting facts about centrally symmetric loops in the plane; for instance, a clean loop with even rotation number $2k$ can never be central unless it passes through its center exactly twice and $k = 0$.

§1 ◇ §2 ◇ §3 ◇ Ref.

1. INTRODUCTION

A set $X \subset \mathbf{R}^{n+1}$ has a *center* $c \in \mathbf{R}^{n+1}$ (or has *central symmetry*, or *is central*) if the c -fixing reflection $x \mapsto 2c - x$ maps X to itself.

What can one say about a set $X \subset \mathbf{R}^{n+1}$ that meets every hyperplane along a central set?

When P is a hyperplane, we (for now) call $X \cap P$ a *cross cut* of X . Later we define cross-cut more narrowly.

Are cross-cuts of central sets always central? Not generally, unless they go through the center. A cube in \mathbf{R}^3 is central, for instance, but a plane that severs its corner cuts it along a triangle, which is never central.

Do central cross-cuts make a set central? Not in \mathbf{R}^2 . For instance, all cross-cuts of a plane triangle are (trivially) central, but again, no triangle is central.

When $n+1 > 2$, however, we know of no such counterexample. Indeed, in the presence of, say, convexity, central cross-cuts can force *more* than just centrality.

For example, when $K \subset \mathbf{R}^{n+1}$ is a convex body and $n + 1 > 2$, central cross-cuts force K to be ellipsoidal. The most general formulation of this fact was proven by S. P. Olovjanischnikoff [O],¹ who relaxed restrictions (e.g., on smoothness) in earlier results of this type by Brunn and Blaschke (see [B1, §44, §84] and [B2]).

In [S1], we drew a similar conclusion for (not necessarily convex) hyper-surfaces of revolution in \mathbf{R}^{n+1} . If their compact convex cross-cuts are central, they must be quadric: ellipsoids, hyperboloids, paraboloids, or circular cylinders.

We later used that fact in [S2] to get a broader result: when a complete immersed surface in \mathbf{R}^3 has a connected compact transverse cross-cut, and all cross-cuts of that type are central, uniformly convex *ovals*, the surface is either a **central cylinder** or a **tubular quadric**.

None of results, however, manages to exploit centrality of cross-cuts without also requiring their convexity. Here for the first time, we drop the convexity requirement, replacing it with an admittedly special but very different alternative. We consider surfaces in \mathbf{R}^3 whose cross-cuts are *clean* (meaning they never visit a point twice tangent to the same line) and have total geodesic curvature zero, making them *figure-8's* up to regular homotopy. Our main result, **Theorem 3.6** says that a surface with this property must be a central cylinder. **Section 3** of our paper focuses on the proof of that fact.

Section 2 (which we find interesting on its own) is devoted mainly the proof of a simple but critical ingredient: *Any clean, central figure-8 must visit its center exactly twice*. The key role this plays in **Section 3** is explained in the paragraphs immediately below the statement of **Theorem 3.6**. In proving the supporting fact, however, we get the general theory summarized in **Proposition 2.16**, which says, in part, that a clean central loop must either have *odd* rotation index, and avoid its center entirely, or else be a figure-8 (index zero) that visits its center exactly twice.

Simple examples—the unit circle traced twice, for instance, or the loop in **Figure 2**—show that such statements fail for loops that are not

¹G. R. Burton gives a nice statement of Olovjanischnikoff's result in [Bu, Lemma 3].

cleanly immersed. The reasoning in both §2 and §3 would simplify considerably if we didn't need to assume and exploit general position arguments to exclude unlikely “pathologies” like these.

2. REPARAMETRIZATION AND SYMMETRY

Definition 2.1 (Central symmetry). An *immersion* $F: M \rightarrow \mathbf{R}^{n+1}$ is central when its image has central symmetry.

Definition 2.2 (Reparametrization). When $\alpha, \beta: \mathbf{S}^1 \rightarrow \mathbf{R}^2$ are immersed loops, we say that β *reparametrizes* α when $\beta = \alpha \circ \phi$ for some diffeomorphism $\phi: \mathbf{S}^1 \rightarrow \mathbf{S}^1$. It *preserves* or *reverses* orientation when ϕ preserves or reverses the orientation of the circle. \square

The following fact will pester us: *Two immersed loops with the same image don't always reparametrize each other, even if they visit each point equally often.* Centrality doesn't mitigate this inconvenient truth, as discussed with regard to Figure 2 below.

Examples 2.3. The unit circle $\mathbf{S}^1 \subset \mathbf{C}$ is central about the origin. If we parametrize it as usual by $t \mapsto e^{it}$, reflection through the origin produces the orientation-*preserving* reparametrization $t \mapsto -e^{it}$.

Contrastingly, consider the figure-8 parametrized by $t \mapsto (\cos t, \sin 2t)$. While likewise central about the origin, reflection through the origin induces the orientation-*reversing* reparametrization $t \mapsto (\cos t, -\sin 2t)$. (Figure 1).

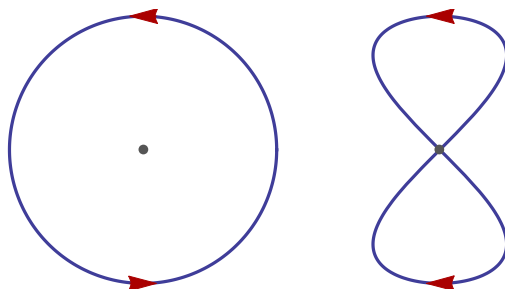


Figure 1. Reflection through the origin reparametrizes both the unit circle and figure-8. While preserving orientation on the circle, however, it *reverses* orientation on the figure-8.

In [Figure 2](#) however, we depict a smooth, origin-central immersion $\alpha: \mathbf{S}^1 \rightarrow \mathbf{R}^2$ whose reflection $-\alpha$ does *not* reparametrize α , even though α and $-\alpha$ have the same image. To see this, orient the open sets U_L, U_0 and U_R there in the standard way, and observe that

$$\begin{aligned}\alpha &= \partial(U_0 + U_R - U_L) \\ -\alpha &= \partial(U_0 - U_R + U_L)\end{aligned}$$

As the oriented domains bounded by α and $-\alpha$ are neither equal nor opposite, $-\alpha$ neither preserves, nor reverses the orientation of α . It evidently does not, therefore, reparametrize. \square

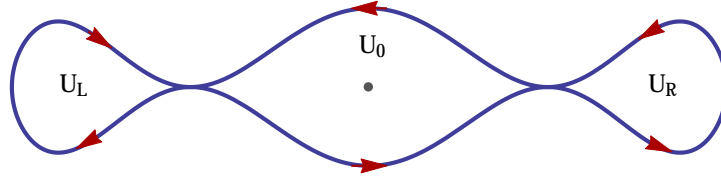


Figure 2. This immersed loop α winds counter-clockwise around U_R and U_0 , but clockwise around U_L . Reflection through the center preserves its image, but $-\alpha$ is not a reparametrization of α .

We can exclude behavior like that depicted in [Figure 2](#) by requiring our loops to be *cleanly immersed*:

Definition 2.4 (Double-points/clean loops). A point p in the image of an immersed curve α is a *double-point* when its preimage contains more than one point. When it contains exactly *two*, we call it a *simple* double-point.

An immersion $\alpha: \mathbf{S}^1 \rightarrow \mathbf{R}^2$ has *clean* double-points (or *is clean*) if, whenever $t_1, t_2 \in \mathbf{S}^1$ are distinct preimages of a single point in \mathbf{R}^2 , they have respective neighborhoods U_1 and U_2 whose images $\alpha(U_1)$ and $\alpha(U_2)$ intersect transversally. \square

Remark 2.5. Though more familiar, a *general position* assumption (like the one we use at the start of [§3](#) below) would be more restrictive than that of *clean double-points* for loops in \mathbf{R}^2 . The latter lets a loop pass through a single point three or more times as long as no two

velocity vectors there are collinear. General position would prohibit triple intersections. \square

The double-points in [Example 2.3](#) are obviously *not* clean, and we shall see that when two loops with the same image do *not* reparametrize each other, they *must* have unclean double-points. Indeed, the principal result of this section, [Proposition 2.11](#), and the main facts leading up to it all fail without cleanness, as consideration of [Figure 2](#) quickly reveals.

We will denote the intrinsic distance between points s_1, s_2 in the sphere of any dimension (here the circle) by $\phi(s_1, s_2)$. We write κ_g for the *geodesic* (or *signed*) *curvature* along a loop $\alpha: \mathbf{S}^1 \rightarrow \mathbf{R}^2$. It is given by

$$(2.1) \quad \kappa_g(t) := \frac{\det(\alpha', \alpha'')}{|\alpha'|^3}$$

Observation 2.6. *Suppose we have a C^2 unit-speed loop $\alpha: \mathbf{S}^1 \rightarrow \mathbf{R}^2$. If $\bar{\kappa} := \max_{\mathbf{S}^1} |\kappa_g|$, and $\alpha^{-1}(p)$ contains distinct inputs $s_1, s_2 \in \mathbf{S}^1$ for some $p \in \mathbf{R}^2$, then $\phi(s_1, s_2) \geq \pi/\bar{\kappa}$.*

Proof. In either arc A joining α_1 to α_2 in \mathbf{S}^1 , some point $S \in A$ maximizes $|\alpha(s) - p|^2$ among $s \in A$, and $\dot{\alpha}(S)$ is then *perpendicular* to $\alpha(S) - p$. At the same time, any non-trivial linear function that vanishes on $\alpha(S) - p$ will attain at least one local extremum on each component of $A \setminus \{S\}$, say at points $\alpha(s_-)$ and $\alpha(s_+)$ respectively. Since $\dot{\alpha}$ must be *parallel* to $\alpha(S) - p$ at these points, the intervals (s_-, S) , $(S, s_+) \subset A$ both map to arcs with total absolute curvature at least $\pi/2$. As α has unit speed, we may then deduce

$$\theta(s_1, s_2) \bar{\kappa} \geq \theta(s_-, s_+) \bar{\kappa} \geq \int_{s_-}^{s_+} |\kappa_g(s)| ds \geq \pi$$

\square

In general, a C^2 immersion $\mathbf{S}^1 \rightarrow \mathbf{R}^2$ can have infinitely many double points, even without retracing any open arc along its image. Not so for clean immersions:

Lemma 2.7. *A clean C^2 immersion $\alpha: \mathbf{S}^1 \rightarrow \mathbf{R}^2$ has at most finitely many double-points, and at any double-point p , there is an $\varepsilon = \varepsilon(p) > 0$ for which $\alpha^{-1}(B(p, \varepsilon))$ is a finite union of embedded arcs passing through p with pairwise distinct tangent lines.*

Proof. With no loss of generality, assume α has unit speed. Set $\bar{\kappa} := \max_{\mathbf{S}^1} |\kappa_g|$ as in the [Observation](#) above.

Suppose (toward a contradiction) that α had infinitely many double points. Since \mathbf{S}^1 and $\alpha(\mathbf{S}^1)$ are both compact, that would imply the existence of a cluster point $p \in \alpha(\mathbf{S}^1)$, along with convergent sequences $(s_n), (s'_n) \subset \mathbf{S}^1$ with

$$s_n \neq s'_n \quad \text{and} \quad \alpha(s_n) = \alpha(s'_n) \rightarrow p$$

and yet $\alpha(s_n) \neq p$ for all $n \in \mathbf{N}$.

[Observation 2.6](#) ensures $|s_n - s'_n| > \pi/\bar{\kappa}$, so the respective limits s and s' of these sequences must obey that same estimate. In particular, $s \neq s'$. By continuity, however, $\alpha(s) = \alpha(s')$, which forces the collinearity of

$$\frac{\alpha(s_n) - \alpha(s)}{s_n - s} \quad \text{and} \quad \frac{\alpha(s'_n) - \alpha(s')}{s'_n - s'}$$

for each n . Letting $n \rightarrow \infty$, we see that $\dot{\alpha}(s)$ and $\dot{\alpha}(s')$ must also be collinear. This contradicts our assumption of clean double-points. So α has at most finitely many double-points.

To prove the remaining claim, we note that since [Observation 2.6](#) puts a lower bound on the distance between any two points in $\alpha^{-1}(p)$, the compactness of \mathbf{S}^1 makes $\alpha^{-1}(p)$ finite. By definition of *immersion*, the inverse function theorem then yields the asserted $\varepsilon(p) > 0$, while that of *clean* makes tangent lines pairwise distinct at p . \square

Lemma 2.8. *Suppose $\alpha, \beta: \mathbf{S}^1 \rightarrow \mathbf{R}^2$ are clean unit-speed C^2 loops with the same image. Suppose $p = \beta(t_0)$ is a double-point of α , and $\varepsilon > 0$ is small enough to make $\alpha^{-1}(B(p, \varepsilon))$ a union of embedded arcs with distinct tangent lines at p , as provided by [Lemma 2.7](#). Then one such arc contains $\beta(t_0 - \delta, t_0 + \delta)$ for all sufficiently small $\delta > 0$.*

Proof. Take $\varepsilon > 0$ small enough to satisfy the hypothesis of [Lemma 2.7](#), and let A_1, A_2, \dots, A_k denote the (distinct) arcs whose union then

constitutes $\alpha^{-1}(B(p, \varepsilon))$. Define $I_n := (t_0 - \frac{1}{n}, t_0 + \frac{1}{n})$ for $n \in \mathbf{N}$. When n is large, β embeds I_n , and since β and α have the same image, $\beta(I_n)$ must then lie in the *union* of the A_i 's.

If for every such n , we could find $t_n, t'_n \neq t$ in I_n with $\beta(t_n)$ and $\beta(t'_n)$ in *different* A_i 's, we could renumber the A_i 's and pass to a subsequence to arrange $\beta(t_n) \in A_1$ and $\beta(t'_n) \in A_2$ for all large n . But $\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} t'_n = t_0$, and β is differentiable, so computing $\dot{\beta}(t_0)$ on the two different sequences would give the same result, forcing the tangent lines to A_1 and A_2 at $p = \beta(t_0)$ to agree. This would contradict the last assertion of [Lemma 2.7](#). So $\beta(I_n)$ must stay in one A_i , as claimed. \square

Definition 2.9. By the *lift* of an immersed unit-speed arc $\alpha: (a, b) \rightarrow \mathbf{R}^2$, we mean the arc parametrized by $s \mapsto (\alpha(s), \dot{\alpha}(s))$ in the unit tangent bundle $\mathbf{R}^2 \times \mathbf{S}^1$. \square

Using [Lemma 2.7](#), the reader will easily verify

Observation 2.10. *If $\alpha: \mathbf{S}^1 \rightarrow \mathbf{R}^2$ is a cleanly immersed loop, its lift is embedded. The lift of any reparametrization $\alpha \circ \phi$ either reparametrizes that of α , or never meets it, depending on whether ϕ preserves or reverses orientation respectively.*

We can now prove the fact that makes the main results of this section accessible.

Proposition 2.11. *Suppose $\alpha, \beta: \mathbf{S}^1 \rightarrow \mathbf{R}^2$ are clean, unit-speed C^2 immersions with the same image. Then β reparametrizes α , and the two loops have the same orientation if and only if their lifts meet.*

Proof. By [Observation 2.10](#), α and β have embedded lifts. If they meet above $\beta(b)$ for some $b \in \mathbf{S}^1$, then [Lemma 2.8](#) provides an $a \in \mathbf{S}^1$ and a $\delta > 0$ such that $(a - \delta, a + \delta)$ and $(b - \delta, b + \delta)$ lift, via α and β respectively, to the same arc in $\mathbf{R}^2 \times \mathbf{S}^1$. The lifts of α and β therefore meet along a set relatively open in the image of each. The coincidence set is also closed (trivially) so the two lifts coincide entirely, manifesting (via the Inverse Function Theorem) a C^1 transition diffeomorphism between them.

The identity map on $\mathbf{R}^2 \times \mathbf{S}^1$ then induces a diffeomorphism between the circles parametrizing α and β , allowing us to read β as a reparametrization of α . Orientation is preserved, for the lifts would otherwise be completely disjoint by [Observation 2.10](#).

If the lifts *are* completely disjoint, then (since clean immersions have at most finitely many double-points by [Lemma 2.7](#)) we can find a point $p \in \alpha(\mathbf{S}^1)$ with a single pre-image $\{t\} = \alpha^{-1}(p)$. Then α and β share a unique tangent line at p . If their lifts don't meet, β must lift to $(p, -\dot{\alpha}(t_0))$ above p . The lift of any orientation-reversing reparametrization β' of β thus *meets* that of α above p , making β' an orientation-preserving reparametrization of α by what we have already proven. So β *reverses* orientation, as claimed. \square

We will apply the [Proposition](#) just proven mainly via this immediate

Corollary 2.12. *Any clean central C^2 loop is reparametrized by its central symmetry.*

As [Figure 1](#) shows, the reparametrization induced by a central symmetry of a clean loop may preserve or reverse orientation. The two possibilities have starkly different geometric implications, however. To see that, we will need [Corollary 2.14](#) below—a further consequence of [Proposition 2.11](#)—which requires this

Definition 2.13. The *centroid* (center of mass) $\mu(\alpha)$ of a C^1 loop $\alpha: \mathbf{S}^1 \rightarrow \mathbf{R}^{n+1}$ with length L is the mean value of α relative to an arc-length parameter s :

$$\mu(\alpha) := \frac{1}{L} \int_{\mathbf{S}^1} \alpha(s) \, ds$$

Note that *the centroid of a loop with central symmetry may not coincide with its center of symmetry*. For example, take the circles $(x \pm 1)^2 + y^2 = 1$, and parametrize their union, starting at $\mathbf{0}$, by tracing clockwise around the right-hand lobe, then counterclockwise around the left, and finally, clockwise around the right again. The origin will be a center of symmetry, but the centroid lies at $(1/3, 0)$. Clean loops, however, never exhibit this kind of discrepancy:

Corollary 2.14. *For a clean, central C^2 loop, center of symmetry and centroid coincide.*

Proof. Suppose $\alpha: \mathbf{S}^1 \rightarrow \mathbf{R}^2$ is a clean C^2 loop with center of symmetry at $\mathbf{c} \in \mathbf{R}^2$, and length L . View it as a unit-speed L -periodic immersion $\mathbf{R} \rightarrow \mathbf{R}^2$. **Proposition 2.11** provides a diffeomorphism ϕ of the circle (which lifts to \mathbf{R}) such that $2\mathbf{c} - \alpha = \alpha \circ \phi$. By the chain rule and constancy of speed (which is preserved by the reflection), we must also have $|\phi'| \equiv 1$. If we denote the unit-speed parameter for α by s , then $u = \phi(s)$ gives a unit-speed parameter for its reflection $2\mathbf{c} - \alpha$, whose centroid is then clearly

$$\begin{aligned} 2\mathbf{c} - \mu(\alpha) &= \frac{1}{L} \int_0^L 2\mathbf{c} - \alpha(s) \, ds \\ &= \frac{1}{L} \int_0^L \alpha \circ \phi(s) \, ds \\ &= \frac{1}{L} \int_0^L \alpha \circ \phi(s) |\phi'(s)| \, ds \\ &= \frac{1}{L} \int_0^L \alpha(u) \, du \\ &= \mu(\alpha) \end{aligned}$$

Thus $\mu(\alpha) = \mathbf{c}$, as claimed. \square

Definition 2.15. When an immersed C^1 loop $\alpha: \mathbf{S}^1 \rightarrow \mathbf{R}^2$ is central with respect to $\mathbf{c} \in \mathbf{R}^2$, we call the line segment joining $\alpha(t)$ to $2\mathbf{c} - \alpha(t)$ a *diameter* of α . If we can parametrize α so that

$$(2.2) \quad 2\mathbf{c} - \alpha(t) = \alpha(t + \pi)$$

for all $t \in \mathbf{S}^1$ (intertwining reflection through \mathbf{c} with the antipodal map on \mathbf{S}^1) we say that α is *diameter-central*.

diameter-central loops are obviously central, but the converse is false, as shown by the central figure-8 in **Figure 1**. Careful consideration of that picture reveals that the figure-8 is not diameter-central. \square

Proposition 2.16. *Suppose $\alpha: \mathbf{S}^1 \rightarrow \mathbf{R}^2$ is a clean, central C^2 loop. Then either*

- a) *the symmetry preserves orientation, in which case α*
 - *is regularly homotopic to $e^{(2k+1)\theta}$ for some $k \in \mathbf{Z}$*
 - *avoids its center, and*
 - *is diameter-central.*

or

- b) *the symmetry reverses orientation, in which case α*
 - *is regularly homotopic to the figure-8,*
 - *has a simple double-point at its center, and*
 - *is not diameter-central.*

Proof. We can assume α is centered at the origin $\mathbf{0}$, and (after a homothety giving it length 2π) has unit speed. **Corollary 2.12** then gives $-\alpha = \alpha \circ \phi$ for some diffeomorphism $\phi: \mathbf{S}^1 \rightarrow \mathbf{S}^1$. By the chain rule, our unit speed assumption forces $|\phi'| \equiv 1$, making ϕ an *isometry* of \mathbf{S}^1 . An isometry either rotates \mathbf{S}^1 or reflects it across a diameter, preserving or reversing orientation respectively.

When we view $\mathbf{S}^1 \approx \mathbf{R}/2\pi$ as an additive group, rotation takes the form $\phi(t) = t + l$ for some $l \in \mathbf{S}^1$. So if the symmetry preserves orientation, we get $-\alpha(t) = \alpha(t + l)$ for all t . Since α is not constant, we may assume $0 < |l| \leq \pi$. Iterating the symmetry then gives $\alpha(t + 2l) = \alpha(t)$, and hence $\dot{\alpha}(t + 2l) = \dot{\alpha}(t)$. Having clean double points, however, obstructs this pair of identities for any $0 < |l| < \pi$. So in the orientation-preserving case, we must have $|l| = \pi$, which makes α diameter-central, as conclusion (a) asserts.

A diameter-central loop has parallel tangent lines at $\alpha(t + \pi)$ and $\alpha(t)$, as follows from differentiating (2.2). Since we assume clean double-points, this forces $\alpha(t + \pi) \neq \alpha(t)$ for all $t \in \mathbf{S}^1$. But we just saw that $\alpha(t + \pi) = -\alpha(t)$ for all $t \in \mathbf{S}^1$. So in the orientation-preserving case, our loop must avoid the origin—its center—as claimed by (a).

In the orientation-reversing case, by contrast, α is reparametrized by an isometry $\phi: \mathbf{S}^1 \rightarrow \mathbf{S}^1$ that reflects across a diameter, fixing two

antipodal points that we can assume, after a rotation, to be $t = 0$ and $t = \pi$. In this case, for all $t \in \mathbf{S}^1$, we have $\phi(t) = -t$, and thus

$$(2.3) \quad -\alpha(t) = \alpha(-t)$$

A central symmetry fixes *only* its center, however, forcing α to map both $t = 0$ and $t = \pi$ to the origin. In fact, the origin must be a *simple* double-point, as (b) claims. For, any central loop has parallel tangent lines at the ends of diameters, and when (2.3) holds, that means parallel tangent lines at $\alpha(t)$ and $\alpha(-t)$ for every $t \in \mathbf{S}^1$. If we had $\alpha(t) = \alpha(-t)$ for some t *not* fixed by ϕ , we would breach our clean double-points assumption.

It remains to verify the claims about regular homotopy. As is well-known, (e.g., [W] or [K1, Proposition 2.1.6]) the regular homotopy class of an immersed plane loop $\alpha: \mathbf{S}^1 \rightarrow \mathbf{R}^2$ is classified by its *rotation index*—the degree $\omega_\alpha \in \mathbf{Z}$ of its unit tangent map $\alpha'/|\alpha'|: \mathbf{S}^1 \rightarrow \mathbf{S}^1$, which we may compute by integrating the geodesic curvature (2.1) along α :

$$(2.4) \quad \omega_\alpha = \frac{1}{2\pi} \int_0^{2\pi} \kappa_g(t) dt,$$

Consider first the orientation-preserving case. There, as we have seen, α is diameter-central: $\alpha(t + \pi) = -\alpha(t)$ for all t . It follows trivially that velocity and acceleration change sign too when we rotate the input by π . As easily seen from formula (2.1), however, this makes κ_g *even* on the circle: $\kappa_g(t + \pi) = \kappa_g(t)$. So when orientation is preserved, the total signed curvature of α is twice that along the arc $\alpha(0, \pi)$. At the same time, we have $\dot{\alpha}(\pi) = -\dot{\alpha}(0)$, forcing the unit tangent $\dot{\alpha}/|\dot{\alpha}|$ to traverse an odd number of semicircles as t varies from 0 to π . So

$$\int_0^{2\pi} \kappa_g(t) dt = 2 \int_0^\pi \kappa_g(t) dt = 2(2k + 1)\pi \quad \text{for some } k \in \mathbf{Z}.$$

By (2.4), we then have $\omega_\alpha = 2k + 1$, an odd integer, as claimed.

In the orientation-reversing case, identity (2.3) replaces the diameter-central condition above. Differentiate that identity twice and use (2.4) to see that κ is now an *odd* function on the circle:

$$\kappa(-t) = -\kappa(t).$$

The integral of an odd function vanishes, so (2.4) now yields $\omega_\alpha = 0$, making α regularly homotopic to the figure-8, as stated. This completes our proof. \square

Corollary 2.17. *A clean C^2 plane loop with even, non-zero rotation index cannot have central symmetry.*

3. MAIN RESULT

Supposing M^n is a smooth manifold, we now take up our motivating question: *What can we say about a complete, proper immersion $F: M^n \rightarrow \mathbf{R}^{n+1}$ when $F(M)$ has central intersections with an open set of hyperplanes?*

To address this, we introduce some notation. We write u_p^\perp for the hyperplane containing $p \in \mathbf{R}^{n+1}$ and normal to $u \in \mathbf{S}^n$. When p is the origin, we simply write u^\perp . These hyperplanes are, respectively, zero sets of the affine functions u_p^* and u^* given by

$$u_p^*(x) = u \cdot (x - p), \quad u^*(x) = u \cdot x$$

When using this notation, we always assume u to be a *unit* vector. We denote the angular distance between unit vectors $u, v \in \mathbf{S}^n$ by $\phi(u, v) := \arccos(u \cdot v)$.

When $a > 0$ and $P = u_p^\perp$, we write P_a for the a -neighborhood of the hyperplane P :

$$(3.1) \quad P_a := \{q \in \mathbf{R}^{n+1} : |u_p^*(q)| < a\}$$

We call $\nu \in \mathbf{S}^n$ a *unit normal* to an immersion $F: M^n \rightarrow \mathbf{R}^{n+1}$ at a point $x \in M$ if ν is orthogonal to the hyperplane $dF(T_x M)$ in \mathbf{R}^{n+1} .

We can then say that F has *general position* if, whenever $y \in \mathbf{R}^{n+1}$ and $\nu_1, \nu_2, \dots, \nu_k$ are unit normals to F at distinct points in $F^{-1}(y)$, we have

$$(3.2) \quad \nu_1 \wedge \nu_2 \wedge \dots \wedge \nu_k \neq 0$$

If this holds when we extend F to $M \cup P$ for some hyperplane $P \subset \mathbf{R}^{n+1}$ via the inclusion map on P , we say that F and P are in *general*

position. Note that in this case, the restriction of F to M must itself have general position.

When (3.2) holds for $k = 2$ (i.e., whenever ν_1, ν_2 are unit normals to F at distinct points of $F^{-1}(y)$), we get weaker conditions that we respectively express by saying F has *transverse self-intersections*, or P meets F *transversally*.

Transversality alone makes $F^{-1}(P)$ an embedded hypersurface in M [H, p.22]. General position guarantees more: when $n = 2$, for instance, it is not hard to see that it makes all double-points of $P \cap F(M)$ *clean* as specified in Definition 2.4.

We want to focus on the case where F and P have general position and the *compact* components of $F^{-1}(P)$ map to sets with central symmetry. Two definitions will help:

Definition 3.1 (Cross-cut). When a hyperplane $P \subset \mathbf{R}^{n+1}$ meets an immersion $F: M^n \rightarrow \mathbf{R}^{n+1}$ transversally, a *cross-cut of F relative to P* is a compact component $\Gamma \subset F^{-1}(P)$. We also call its image $F(\Gamma)$ a cross-cut; context will signal which meaning applies.

We call Γ a *clean* cross-cut when P and F are in general position. \square

The transversality assumption in Definition 3.1 ensures that the tangential projection $u \mapsto u - (u \cdot \nu)\nu$ yields a *non-vanishing* transverse vectorfield along Γ (the choice of unit normal ν to F is obviously irrelevant here). Cross-cuts are thus *orientable* in M . A routine differential topology exercise then yields the existence of what we shall call a *good tubular coordinate neighborhood* U of a cross-cut Γ . This is a neighborhood that F maps to a tube foliated by cross-cuts diffeomorphic to Γ , each a level set of the height function u_p^* .

Definition 3.2 (Good tubular patch). Suppose $\Gamma \subset M$ is a cross-cut of F relative to a hyperplane $P = u_p^\perp$. By a *good tubular coordinate neighborhood* (or *good tubular patch*) for Γ , we mean a pair (U, ψ) , where $U \subset M$ is the image of an embedding $\psi: \Gamma \times [-a, a] \rightarrow M$ for some $a > 0$, and ψ has these three properties for all $(\theta, h) \in \Gamma \times [-a, a]$:

- a) $\psi(x, 0) = x$ for all $x \in \Gamma$

b) $(u_p^* \circ F \circ \psi)(\theta, h) = h$ and

c) $d(u_p^* \circ F \circ \psi) \neq 0$

Property (b) means that for each $h \in [-a, a]$, the composition $F \circ \psi$ maps $\Gamma \times \{h\}$ into the plane $u_p^* \equiv h$. Property (c) makes F transverse to these same planes, so that $\psi(\Gamma \times \{h\})$ is a cross-cut of F for each $h \in [-a, a]$. \square

As mentioned above, the *existence* of a good tubular neighborhood of a cross-cut is easy to establish. When a cross-cut is *clean*, we can guarantee that nearby cross-cuts are likewise clean:

Lemma 3.3. *Suppose $\Gamma \subset M$ is a clean cross-cut relative to $P = u_p^\perp$, and (U, ψ) is a good tubular patch for Γ . Then there is an $\varepsilon > 0$ for which $|q - p| < \varepsilon$ and $\phi(v, u) < \varepsilon$ together ensure that $F^{-1}(v_q^\perp) \cap U$ is again a clean cross-cut, and is regularly homotopic to Γ .*

Proof. Define the map

$$\mathcal{F}: U \times \mathbf{R}^{n+1} \times \mathbf{S}^n \rightarrow \mathbf{R} \times \mathbf{R}^{n+1} \times \mathbf{S}^n$$

via

$$\mathcal{F}(x, q, v) = ((F(x) - q) \cdot v, q, v)$$

Property (c) in [Definition 3.2](#) makes $d\mathcal{F}$ surjective at each point of $\mathcal{F}^{-1}(0, p, u) = \Gamma \times \{p\} \times \{u\}$, and lower-semicontinuity of rank then makes $d\mathcal{F}$ surjective on some neighborhood of $\Gamma \times p \times u$. If we denote ε -neighborhoods of p and u in \mathbf{R}^{n+1} and \mathbf{S}^n respectively by $B_\varepsilon(p)$ and $B_\varepsilon(u)$, the Implicit Function Theorem and compactness of Γ then make it straightforward to deduce that for some $\varepsilon > 0$, the \mathcal{F} -preimage of $(-\varepsilon, \varepsilon) \times B_\varepsilon(p) \times B_\varepsilon(u)$ is foliated by preimages $\mathcal{F}^{-1}(h, q, v)$, all regularly homotopic to $\mathcal{F}^{-1}(0, p, u) = \Gamma \times \{p\} \times \{u\}$. It follows that $\mathcal{F}^{-1}(\{0\} \times B_\varepsilon(p) \times B_\varepsilon(u))$ is likewise foliated. Since

$$\mathcal{F}^{-1}(0, q, v) = (F^{-1}(v_q^\perp) \cap U) \times \{q\} \times \{v\}$$

this shows that $|q - p| < \varepsilon$ and $\phi(v, u) < \varepsilon$ together ensure, for every such q and v , that $F^{-1}(v_q^\perp) \cap U$ is a cross-cut regularly homotopic to Γ .

Finally, by making $\varepsilon > 0$ smaller still if necessary, we can guarantee that these cross-cuts are all *clean* too. Otherwise, we could find convergent sequences $(q_k) \rightarrow p$ and $(v_k) \rightarrow u$ for which each corresponding cross-cut $(F(x) - v_k) \cdot q_k \equiv 0$ in U was *not* clean. Condition (3.2) would then have to fail at some point y_k in each of these cross-cuts. Condition (3.2) is continuous in all variables, however, F is C^1 , and U is compact. Passing to a subsequence, we could then take a limit as $k \rightarrow \infty$ and force a contradiction to our assumption that Γ itself was clean. \square

With these purely differential-topological facts in hand, we now turn the case of interest: where (the images of) all cross-cuts have central symmetry.

Definition 3.4 (CX). An immersion $F: M \rightarrow \mathbf{R}^{n+1}$ has the *central cross-cut property* (abbreviated CX) when

- a) At least one clean cross-cut exists, and
- b) The image of every clean cross-cut has *central symmetry*.

Note that CX is an affine-invariant property: if F has CX, and A is an affine isomorphism of \mathbf{R}^{n+1} , then $A \circ F$ has CX too. \square

In \mathbf{R}^3 , circular cylinders and spheres have CX, and they represent the only two kinds of examples we know:

- *Central cylinders:* If an immersion with a cross-cut is preserved by a line of translations *and* by a central reflection, we call it a *central cylinder*. Central cylinders clearly have CX, since every cross-cut is a translate of one through the center.
- *Tubular quadrics:* When a non-degenerate quadric hypersurface in \mathbf{R}^{n+1} is affinely equivalent to a locus of the form

$$x_1^2 + x_2^2 + \cdots x_n^2 \pm x_{n+1}^2 = c \in \mathbf{R}$$

it will always have compact and transverse, hence ellipsoidal (and thus central) cross-cuts. We call these hypersurfaces *tubular quadrics*. Note that in \mathbf{R}^3 , *all* non-degenerate quadrics are tubular.

We suspect these two classes exhaust all possibilities:

Conjecture 3.5. *A complete immersion $F: M^n \rightarrow \mathbf{R}^{n+1}$ with CX must either be a central cylinder, or a tubular quadric.*

In previous papers, we confirmed weakened versions of this conjecture, proving it

- for C^1 hypersurfaces of revolution ($\mathrm{SO}(n)$ symmetry) in \mathbf{R}^{n+1} [S1], and then, using that result,
- for C^2 surfaces in \mathbf{R}^3 whose cross-cuts are *convex* as well as central [S2].

Here we add another case to this list: roughly, that of a complete surface in \mathbf{R}^3 with CX *and* for which some clean cross-cut is a figure-8.

To make this precise, we first note that on any complete immersed C^2 surface with CX in \mathbf{R}^3 , every clean cross-cut is a (clean) central C^2 plane loop. By Proposition 2.16, then, each of these loops is either regularly homotopic to a figure-8, *or* has odd rotation index.

The rotation index of a figure-8 is zero, and here (as sketched in our introduction) we verify Conjecture 3.5 for immersions with figure-8 cross-cuts. Since cross-cuts of quadrics can't be figure-8's, such immersions must be cylindrical:

Theorem 3.6 (Main Result). *If $F: M \rightarrow \mathbf{R}^3$ is a complete C^2 immersion with CX, and some plane in general position relative to F cuts it along a clean figure-8, then $F(M)$ is a central cylinder.*

The figure-8 assumption is decisive for the following reason: When a plane P cuts a surface with CX transversally along a figure-8 centered at $c \in \mathbf{R}^3$, and we tilt P slightly about c to get nearby cross-cuts, *the latter remain centered at c .*

Without the figure-8 assumption, this fails.

Indeed, consider the unit sphere $\mathbf{S}^2 \subset \mathbf{R}^3$. It clearly has CX. Now take $u \in \mathbf{S}^2$, $0 < \lambda < 1$, and set $c := \lambda u$. The plane u_c^\perp will cut \mathbf{S}^2 along a circle centered at c . For any $v \in \mathbf{S}^2$ near u , however, the cross-cut $v_c^\perp \cap \mathbf{S}^2$ is clearly centered on the line spanned by λv (Figure 3). So for $v \neq u$, the center moves.

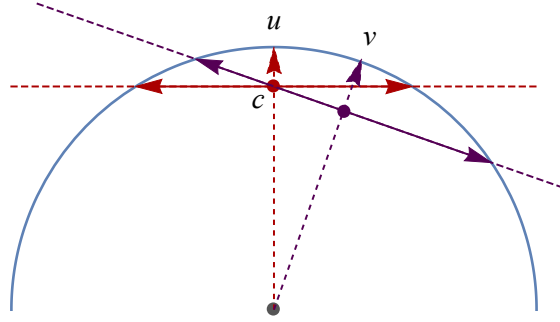


Figure 3. Cross-cuts on a sphere, via u_c^\perp and v_c^\perp . Both hyperplanes *contain* c , but only one of the cross-cuts (red) is centered at c .

The center *cannot* move in this way when cross-cuts are figure-8's, as we make precise in [Lemma 3.8](#) shortly below, using the notion of *central curve* of a good tubular patch:

Definition 3.7 (Central curve). Suppose Γ is a cross-cut for an immersion $F: M \rightarrow \mathbf{R}^{n+1}$ relative to a hyperplane $P = u_p^\perp$. Let (U, ψ) denote a good tubular patch for Γ as in [Definition 3.2](#), so that F maps $\psi(\Gamma, h)$ into the hyperplane $u_c^* \equiv h$ for each $h \in [-a, a]$. The *central curve* of the patch is the map $\mu: [-a, a] \rightarrow \mathbf{R}^{n+1}$ sending any $h \in [-a, a]$ to the centroid $\mu(h)$ of $F(\psi(\Gamma, h))$. \square

When F is C^k , the central curve of a good tubular patch is clearly C^k too. It is also *immersed*, since condition (b) from [Definition 3.2](#) yields $u_p^*(\mu(h)) = h$, and hence $\dot{\mu}(h) \cdot u \geq 1$.

In the Lemma below, we formulate the advantage offered by figure-8 cross-cuts. Notation is as above: $F: M^2 \rightarrow \mathbf{R}^3$ is a proper C^2 immersion, $u \in \mathbf{S}^2$ and $c \in \mathbf{R}^3$ are fixed. We have a clean cross-cut $\Gamma \subset F^{-1}(u_c^\perp)$, for which $U \subset M$ is a good tubular neighborhood ([Definition 3.2](#)), and $\mu: [-a, a] \rightarrow \mathbf{R}^3$ is its central curve.

Lemma 3.8. *Suppose F has CX, $F(\Gamma)$ is a figure-8, and $\varepsilon > 0$. If $\Gamma_{h,v} := U \cap F^{-1}(v_{\mu(h)}^\perp)$ is a clean cross-cut, regularly homotopic to Γ whenever $|h| < \varepsilon$ and $v \in \mathbf{S}^2$ with $\phi(v, u) < \varepsilon$, then $F(\Gamma_{h,v})$ is a figure-8 with central symmetry about $\mu(h)$ for all such h and v .*

Proof. **Lemma 3.3** says that for all sufficiently small $|h|$, the cross-cut $\Gamma_{h,u}$ (cut by the plane at signed height h above u_c^\perp) is, like $\Gamma_{0,u} = \Gamma$ itself, clean and regularly homotopic to Γ . For simplicity, we can assume this holds for all $|h| \leq a$. (If not, re-define our good tubular patch using a smaller $a > 0$.)

In this case, $F(\Gamma_{h,u})$ is a clean figure-8 for every $|h| \leq a$, and its center, by **Proposition 2.16(b)**, is a simple double-point. The central curve μ of the patch thus consists entirely of simple double-points.

In particular, if we fix any $h \in (-\varepsilon, \varepsilon)$, then $F^{-1}(\mu(h)) \cap U$ is a pair $\{x_1, x_2\}$, and as an immersion, F embeds disjoint neighborhoods $U_1 \supset x_1$ and $U_2 \supset x_2$ in such a way that, in the ball $B_{h,r}$ centered at $\mu(h)$ with sufficiently small radius $r > 0$, we have

$$F(U) \cap B_{r,h} \subset F(U_1) \cup F(U_2)$$

Further, since F has general position, we can make $r > 0$ small enough to ensure that in $B_{r,h}$, the sheets $F(U_1)$ and $F(U_2)$ meet along a segment of the central curve and nowhere else.

Now, as long as $\phi(v, u) < \varepsilon$, the nearby cross-cut $\Gamma_{h,v}$ is, by assumption, another clean cross-cut in U , regularly homotopic to $\Gamma = \Gamma_{0,u}$. Immersion preserves regular homotopy, so for all such v , the nearby images $F(\Gamma_{h,v})$, like $F(\Gamma)$, are clean figure-8's—and they are central, since F has CX. We just need to show they stay centered, like $F(\Gamma_{h,u})$, at $\mu(h)$.

To see that they are, note that the planes $v_{\mu(h)}^\perp$ all cut the central curve μ transversally at $\mu(h)$ since the cross-cuts they form are all clean. So by shrinking $r > 0$ further if needed, we can ensure that in $B_{r,h}$, each of these planes cuts the central curve *only* at $\mu(h)$.

It follows that $\mu(h)$ is the *unique* double-point that $F(\Gamma_{h,v})$ has in $B_{r,h}$. Since $\Gamma_{h,v}$ varies smoothly with v , its centroid—and center of symmetry by **Corollary 2.14**—varies smoothly too. So for v sufficiently near u , the center of the figure-8 $F(\Gamma_{h,v})$ must stay in $B_{r,h}$. As seen above, however, that center is a simple double-point, and we have just noted that for every v in question, the *only* double-point of $F(\Gamma_{h,v})$ in $B_{r,h}$ is $\mu(h)$. When $\phi(v, u) < \varepsilon$, the center of $F(\Gamma_{h,v})$ is therefore

trapped at $\mu(h)$. As this holds for whenever $|h| < \varepsilon$, we have proven the Lemma. \square

We will prove our main result ([Theorem 3.6](#)) by combining this Lemma with the [Local Axis Lemma](#) below, which shows that when F has CX, and centers of tilted cross-cuts stay (locally) on the central curve as in the Lemma above, the central curve is locally straight. Note that it makes no figure-8 assumption. This lemma quickly produces a local version of the main result, namely [Corollary 3.10](#).

As above, we write $a > 0$ and $P = u_c^\perp$ for a fixed (but arbitrary) scalar and plane respectively; P_a denotes the a -neighborhood of P . We have a clean cross-cut $\Gamma \subset F^{-1}(P)$, and a good tubular patch $\psi: \Gamma \times [-a, a] \rightarrow U \subset M$ around Γ , so that $F(\partial U) \subset \partial P_a$. Without loss of generality, we assume $c = \mu(0)$, the initial value of the central curve μ of $F(U)$.

Lemma 3.9 (Local axis lemma). *Suppose $\varepsilon > 0$, $0 < b < a$ and $F^{-1}(v_{\mu(t)}^\perp) \cap U$ is a boundaryless clean cross-cut whose image is central about $\mu(t)$ whenever $\phi(u, v) < \varepsilon$ and $|t| < b$. Then μ maps $[-b, b]$ to a line segment.*

Proof. We may identify $\Gamma \approx \mathbf{S}^1$, and simplify notation accordingly by using coordinates from the domain of our good tubular patch so that, for instance, $F(\theta, h)$ really means $F(\psi(\theta, h))$.

Fix an arbitrary $\zeta \in (-b, b)$, and choose $\theta_0 \in \mathbf{S}^1$ so that $p_0 := F(\theta_0, \zeta)$ maximizes $|F(\theta, \zeta)|^2$ on $F(\Gamma, \zeta)$:

$$|p_0|^2 = |F(\theta_0, \zeta)|^2 \geq |F(\theta, \zeta)|^2 \quad \text{for all } \theta \in \mathbf{S}^1$$

To prove the Lemma, we will first need to show that $(\theta_0, \zeta) \subset U$ has a neighborhood with certain favorable attributes. For that, note that $|F(\theta, s) - \mu(h)|$ is continuous on the set of triples $(\theta, s, h) \in \Gamma \times [-a, a]^2$, and that $|p_0 - \mu(\zeta)| = 2r$ for some $r > 0$. So by making $\eta > 0$ small enough, we can ensure two properties:

- i) $|\zeta \pm \eta| < b$,
- ii) $|\theta - \theta_0|, |s - \zeta|, |h - \zeta| < \eta \Rightarrow |F(\theta, s) - \mu(h)| > r$

Now for any (θ, s, h) in the η -neighborhood of (θ_0, ζ, ζ) defined by (ii) above, consider the unit vector

$$w = w(\theta, s, h) := \frac{F(\theta, s) - \mu(h)}{|F(\theta, s) - \mu(h)|}$$

Combining (b) from [Definition 3.2](#) with (ii), we then have

$$(3.3) \quad |u \cdot w| = \frac{|s - h|}{|F(\theta, s) - \mu(h)|} \leq \frac{|s - h|}{r}$$

Subtract the w -component from u and normalize to construct a unit vector normal to w :

$$(3.4) \quad v := \frac{u - (u \cdot w)w}{|u - (u \cdot w)w|}$$

By design, the plane $v_{\mu(h)}^\perp$ now contains $F(\theta, s)$. We shall want it to cut $F(U)$ along a central loop, and our hypotheses certify that, *if* we can show $\phi := \phi(u, v) < \varepsilon$. To do so, combine (ii) with the triangle inequality to deduce $|s - h| < 2\eta$, and hence

$$\sin^2 \phi = 1 - \cos^2 \phi = 1 - (u \cdot v)^2 = (u \cdot w)^2 \leq \left| \frac{s - h}{r} \right|^2 \leq \frac{4\eta^2}{r^2}$$

Since $\phi < \varepsilon$ when $\sin \phi < \sin \varepsilon$, this yields the bound we seek if we require, along with (i) and (ii) above, that

$$\text{iii)} \quad 0 < \eta < \frac{r \sin \varepsilon}{2}$$

Together, restrictions (i), (ii), and (iii) on $\eta > 0$ now leverage our hypotheses to ensure that for v given by (3.4), the plane $v_{\mu(h)}^\perp$ contains both $F(\theta, s)$ and $\mu(h)$, and cuts $F(U)$ along a loop with central symmetry about $\mu(h)$.

We can now make the main geometric argument for our lemma.

Consider the mapping that sends (θ, s, h) to the reflection of $F(\theta, s) \in F(U)$ through $\mu(h)$:

$$(3.5) \quad (\theta, s, h) \longmapsto 2\mu(h) - F(\theta, s)$$

Our hypotheses guarantee that for all small enough $|\tau| > 0$, the arc parametrized by

$$\beta(\tau) := 2\mu(h + \tau) - F(\theta, s)$$

stays in $F(U)$. Trivially, its initial velocity is $2\dot{\mu}(h)$, which cannot vanish because $u^*(\dot{\mu}(h)) = 1$, by condition (b) from [Definition 3.2](#). This proves:

If $\eta > 0$ satisfies (i), (ii), and (iii) above, then for all (θ, s, h) with $|\theta - \theta_0|, |s - \zeta|, |h - \zeta| < \eta$, the plane tangent to $F(U)$ at $2\mu(h) - F(\theta, s)$ contains $\dot{\mu}(h) \neq \mathbf{0}$.

It follows immediately that whenever $|t| < \eta$, each tangent plane to $F(U)$ in a neighborhood of $p_0 = F(\theta_0, \zeta)$ contains both $\dot{\mu}(\zeta)$ and $\dot{\mu}(\zeta + t)$. From this, we can deduce constancy of $\dot{\mu}$ near ζ .

Indeed, we would otherwise have $\dot{\mu}(\zeta + t) \neq \dot{\mu}(\zeta)$ for some $t \in (-\eta, \eta)$, and since they have the same u -component, by (b) from [Definition 3.2](#), inequality means independence. Since p_0 has a neighborhood in $F(U)$ where every tangent plane contains—hence is spanned by—these same two non-zero vectors, independence forces constancy of the unit normal to $F(U)$ near p_0 . A neighborhood of p_0 in $F(U)$ then lies in a plane—a plane cutting $u_{p_0}^\perp$ along a line. The cross-cut parametrized by $F(\cdot, \zeta)$ must contain a segment of that line, with $p_0 = F(\theta_0, \zeta)$ in its interior. But we maximized $|F(\theta, \zeta)|^2$ at θ_0 , and $x \mapsto |x|^2$ is strictly convex; it *cannot* reach a local max on the interior of a segment. We have thus contradicted the possibility that $\dot{\mu}(\zeta + t) \neq \dot{\mu}(\zeta)$ for any $|t| < \eta$. It follows that $\dot{\mu} \equiv \dot{\mu}(\zeta)$ on a neighborhood of ζ .

Because $\zeta \in (-b, b)$ was arbitrary, however, this (and continuity of $\dot{\mu}$) yields local constancy of $\dot{\mu}$ on subset of $[-b, b]$ that is simultaneously non-empty, open, and closed. The conclusion of our Lemma follows at once. \square

Corollary 3.10 (Local cylinder). *Under the assumptions of [Lemma 3.9](#), $F(U)$ is a central cylinder.*

Proof. By [Lemma 3.9](#), the central curve of $F(U)$ —what we shall henceforth call its *axis*—is a line segment parallel to $v := \dot{\mu}(0)$. The Corollary follows easily from one additional

Claim. Every tangent plane to $F(U)$ contains v .

As $F(U)$ is closed, it suffices to prove this for an arbitrary point $p \in F(U)$ not lying on its axis. Let $h := u_c^*(p)$ denote the signed height of p above u_c^\perp ($c = \mu(0)$).

The cross-cut of $F(U)$ parallel to u_c^\perp at height h is central about $\mu(h)$, so both p and its reflection $q := 2\mu(h) - p$ lie in $F(U) \cap u_{\mu(h)}^\perp$. Like p , of course, q avoids the axis of $F(U)$.

Our assumptions say that slightly tilted cross-cuts of $F(U)$ are *also* central about the axis, and provided $|t| > 0$ is sufficiently small, some such cross-cut contains both q and $\mu(h+t)$.

It follows that the arc $t \mapsto 2\mu(h+t) - q$ lies in $F(U)$ for all sufficiently small $|t|$. The resulting differentiable arc passes through p when $t = 0$, with initial velocity $2\dot{\mu}(h) = 2v$, so v is tangent to $F(U)$ at p , as our Claim proposes.

The corollary quickly follows: $F(U)$ is everywhere tangent to the constant vectorfield v , so it is foliated by line segments parallel to v . This makes it a (generalized) cylinder, and it has a compact central cross-cut, so it is central too. \square

By chaining together intermediate results from above, we can quickly prove our main theorem. We restate it here for the reader's convenience.

Theorem 3.6: *If $F: M \rightarrow \mathbf{R}^3$ is a complete C^2 immersion with CX, and some plane in general position relative to F cuts it along a clean figure-8, then $F(M)$ is a central cylinder.*

Proof of Main Result. We are assuming that for some plane $P = u_c^\perp$ in general position relative to F , a clean cross-cut $\Gamma \subset F^{-1}(P)$ whose image $\gamma := F(\Gamma) \subset P$ is a clean figure-8. As discussed in connection with Definition 3.2, that puts Γ in the image of a good tubular patch (U, ψ) .

Lemma 3.3 now provides some $0 < \varepsilon < a$ for which every cross-cut $F^{-1}(v_{\mu(h)}^\perp) \cap U$ is a clean figure-8 when $|h| < \varepsilon$ and $\phi(v, u) < \varepsilon$. As above, $\mu: [-a, a] \rightarrow \mathbf{R}^3$ here denotes the central curve of $F(U)$.

Lemma 3.8 now certifies that each of these cross-cuts has central symmetry about $\mu(h)$, and Corollary 3.10 (with $b = \varepsilon$) then shows that,

within the ε -neighborhood P_ε of P , the image $F(U)$ is a central cylinder.

A simple open/closed argument now shows that $F(M)$ is the complete extension of that cylinder.

Indeed, call a scalar $a > 0$ *reachable* if there exists a good tubular patch $\psi: \Gamma \times [-a, a] \rightarrow M$ whose image is mapped by F to a central cylinder. Given what we have just proven, we know that

$$A := \sup \{a > 0 : a \text{ is reachable}\} \geq \varepsilon > 0$$

Our theorem amounts to the assertion $A = \infty$, which we can now establish by contradiction. For if $A < \infty$, the completeness and smoothness of F would let us construct a *maximal* good tubular patch $\psi: \Gamma \times [-A, A] \rightarrow M$, with $F \circ \psi$ mapping $\Gamma \times [-A, A]$ to a central cylinder with boundary in ∂P_A . The two loops bounding this cylinder would clearly be clean figure-8's. By applying the argument above, however, we could deduce that their preimages in M each have good tubular neighborhoods mapping to central cylinders via F . Our supposedly maximal good tubular patch could then be extended slightly at each boundary component, violating the maximality of A . Thus, A *cannot* be finite; we have $A = \infty$ which gives our theorem. \square

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